Probabilistic Inequalities
for Sums of Heavy-Tailed Random Matrices

Stas Minsker (University of Southern California)

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 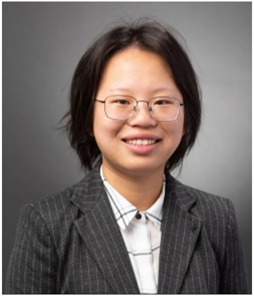 

based on a joint work in with Moritz Jirak, Yiqiu Shen and Martin Wahl

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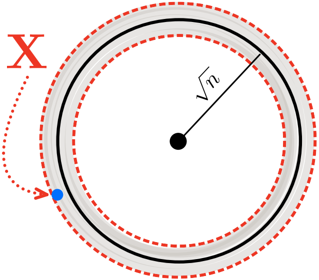
- Examples include the Gaussian (Borell-TIS) inequality, bounded difference (McDiarmid’s) inequality, Talagrand’s inequality for product measures, matrix Bernstein’s inequality, etc.
- For example, if $\mathbf{X} = (X_1, \ldots, X_n) \sim N(0, I_n)$ then $\mathbb{E}\|\mathbf{X}\|_2 \in \left[\frac{n}{\sqrt{n+1}}, \sqrt{n}\right]$ and

$$\Big| \|\mathbf{X}\|_2 - \mathbb{E}\|\mathbf{X}\|_2 \Big| \leq \sqrt{2t}$$

with probability at least $1 - e^{-t}$.



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- If $\mathbf{W} \in \mathbb{R}^{n \times p}$, $n \geq p$, has i.i.d. normal coordinates, then

$$\left\| \frac{W^T W}{n} - I_p \right\| \leq \sqrt{\frac{p}{n}} + \sqrt{\frac{2t}{n}}$$

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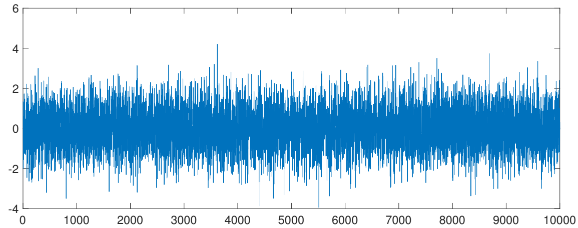
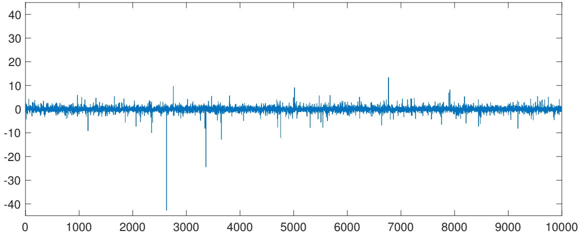
- For the purpose of this talk, a random variable Z has heavy-tailed distribution if

$$\mathbb{E}|Z|^k = \infty$$

  for some $k > 2$.



Standard normal distribution



Student's t-distribution with 3 d.f.

Fuk-Nagaev inequalities

- (Sub)-Gaussian concentration: let $X_1, \ldots, X_n$ be independent, $X_j \sim N(0, \sigma_j^2)$. Then

$$\mathbb{P}\left(\Big| \sum_{j=1}^{n} X_j \Big| \geq t \right) \leq 2 \exp\left(\frac{t^2}{2 \sum_{j=1}^{n} \sigma_j^2} \right)$$

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- Dao Ha Fuk and Sergei Nagaev (1971): let $X_1, \ldots, X_n$ be i.i.d.centered random variables with $p \geq 2$ finite moments. Then

$$\mathbb{P}\left(\left| \sum_{j=1}^{n} X_j \right| \geq t \right) \leq 2 \exp \left(-C_1(p) \frac{t^2}{\sum_{j=1}^{n} \mathbb{E} X_j^2} \right) + \mathbb{P}\left(\max_j |X_j| > t/4 \right)$$
$$+ C_2(p) \left(\frac{\sum_{j=1}^{n} \mathbb{E}|X_j|^p}{t^p} \right)^{2}$$

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- E. Rio (2017), M. Bakhshizadeh, A. Maleki, V. H. de la Pena (2022) proved bounds implying that

$$\mathbb{P}\left(\left| \sum_{j=1}^{n} X_j \right| \geq t \right) \leq \exp\left(-\frac{t^2}{(2+\delta) \sum_{j=1}^{n} \mathbb{E} X_j^2} \right) + C_2(\delta) p^p \frac{\mathbb{E} \max_j |X_j|^p}{t^p}$$

Fuk-Nagaev inequalities

- Uwe Einmahl and Deli Li (2007); Radek Adamczak (2008) proved that for Banach space-valued centered random variables,

$$\mathbb{P}\left(\Big\|\sum_{j=1}^{n} X_j\Big\| \geq (1+\eta)\mathbb{E}\Big\|\sum_{j=1}^{n} X_j\Big\| + t\right) \leq \exp\left(-\frac{t^2}{(2+\delta)B_n^2}\right) + C(\eta,\delta,p)\frac{\sum_{j=1}^{n}\mathbb{E}\|X_j\|^p}{t^p}$$

where $B_n^2 = \sup_{\|f\|_*=1}\sum_{j=1}^{n}\mathbb{E}\left\langle f, X_j\right\rangle^2$.

Fuk-Nagaev inequalities

- Rio's and Einmahl and Li's inequalities can be used to prove versions of the bounded Law of the Iterated Logarithm.

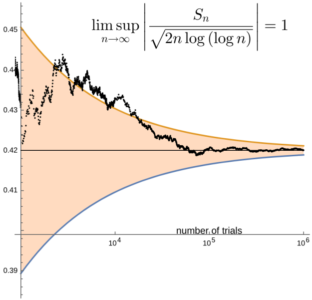

Figure: demonstrations.wolfram.com

Moment inequalities

- Haskell Rosenthal's inequality (1970): let $X_1, \ldots, X_n$ be i.i.d.centered random variables with $p \geq 2$ finite moments. Then

$$\mathbb{E}^{1/p}\Big|\sum_{j=1}^{n} X_j\Big|^p \leq C(p)\left(\left(\sum_{j=1}^{n} \mathbb{E}X_j^2\right)^{1/2} + \left(\sum_{j=1}^{n} \mathbb{E}|X_j|^p\right)^{1/p}\right)$$

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- Johnson, Schechtman and Zinn (1985): best possible $C(p) \asymp \frac{p}{\log(p)}$.

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- Alternatively, one may ask for $C_1(p)$ and $C_2(p)$ such that

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- Up to absolute constants, $(*)$ implies Bernstein's inequality: if $|X_j| \le U$ a.s. for all j, then

$$\mathbb{P}\left(\Big|\sum_{j=1}^{n} X_j\Big| \ge t\right) \le 2\exp\left(-\frac{t^2/2}{\sum_j \mathbb{E}X_j^2 + Ut/3}\right)$$

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- In this talk, we will show that it is possible to have $C_1(p) \asymp \sqrt{p},\ C_2(p) \asymp \frac{p}{\log(p)}$ if

$$\mathbb{E}\max_j |X_j| \lesssim (\log(p))^{-1}\left(\mathbb{E}\max_j |X_j|^p\right)^{1/p}.$$

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- Example: $Y_1, \dots, Y_n \in \mathbb{R}^d$, $\mathbb{E} Y_j = 0$, $\mathbb{E} Y_j Y_j^T = \Sigma$, and $X_j = \frac{1}{n}\left(Y_j Y_j^T - \Sigma\right)$:

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- <u>Covariance estimation:</u> if $\mathbb{E}^{1/p} \max_j \|Y_j\|^{2p} \asymp d\|\Sigma\|$, then $\left\|\widehat{\Sigma}_n - \Sigma\right\| \le \varepsilon\|\Sigma\|$ as long as

$$n \gtrsim \frac{1}{\varepsilon^2} d \log(d).$$

Boundedness and the effective rank

- Covariance estimation: what if Σ can be well approximated by a matrix with small rank? And what if $\|Y_j\|$ is unbounded (but, say, $\mathbb{E}e^{\lambda\|Y_j\|^2} < \infty$ for some $\lambda > 0$)?

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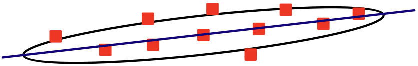
- Effective rank:

$$r(A) = \frac{\mathrm{tr}(A)}{\|A\|}$$



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- Results in this direction were obtained by M. (2011), Koltchinskii (2011), Klochkov and Zhivotovskiy (2018). Assume that $\sum_{j=1}^n \mathbb{E}X_j^2 \preceq V_n$, $B_n^2 = \|V_n\|$, $M := \max_j \|X_j\|$. Then

$$\mathbb{P}\left(\left\|\sum_{j=1}^n X_j\right\| \geq t\right) \leq C_1 \cdot r(V_n) \exp\left(-C_2 \frac{t^2}{B_n^2 + \|M\|_{\psi_1} t}\right)$$

Main results: Fuk-Nagaev-type inequality

Theorem (J+M+S+W)

Let $X_1, \ldots, X_n$ be centered self-adjoint random matrices such that $\mathbb{E}\|X_1\|^p < \infty$ and

$$\sum_{j=1}^{n} \mathbb{E}X_j^2 \preceq V_n, \quad B_n^2 = \|V_n\|, \quad M := \max_j \|X_j\|$$

Then

$$\mathbb{P}\left(\left\|\sum_{j=1}^{n} X_j\right\| \geq t\right) \lesssim r(V_n) \exp\left(-C_2 \frac{t^2}{B_n^2 + t\,\mathbb{E}M}\right) + \mathbb{P}(M \geq t/80)$$

$$+\left(\frac{p}{\log(ep)}\right)^{2p} \left(\frac{\mathbb{E}M^p}{t^p}\right)^2$$

Main results: Rosenthal-type inequality

Integrating the Fuk-Nagaev inequality, we deduce the following

Corollary

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For all $p \geq 1$*,*

$$\mathbb{E}^{1/p} \left\| \sum_{j=1}^{n} X_j \right\|^p \lesssim \sqrt{q} B_n + q\,\mathbb{E} M + \frac{p}{\log(ep)} \left(\mathbb{E} M^p\right)^{1/p} \tag{$*$}$$

where $q = \max(p, \log(r(V_n)))$.

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- Compare to

$$\mathbb{E}^{1/p} \left\| \sum_{j=1}^{n} X_j \right\|^p \lesssim \sqrt{q} B_n + q \left(\mathbb{E}M^p\right)^{1/p},$$

  $q = \max(p, \log(ed))$.
- If $\mathbb{E}M \ll \mathbb{E}^{1/p} M^p$ then $(*)$ improves the scalar version of Rosenthal's inequality.

Applications to covariance estimation

- $\widehat{\Sigma}_n = \frac{1}{n}\sum_{j=1}^{n} Y_j Y_j^T$, and $\left\|\widehat{\Sigma}_n - \Sigma\right\| \le \varepsilon\|\Sigma\|$ as long as

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- What about bounds in L_p? Using Rosenthal's inequality, we prove the following bound:

Corollary

$$\mathbb{E}^{1/2} \left\|\widehat{\Sigma}_n - \Sigma\right\|^2 \lesssim_{p,\kappa} \frac{\left(\mathbb{E} \max_j \|Y_j\|^p\right)^{2/p}}{n} + \|\Sigma\| \sqrt{\frac{r(\Sigma)}{n}}$$

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$$\mathbb{E}^{1/2} \left\| \hat{u}_j - u_j \right\|_2^2 \lesssim_{p,\kappa} \sqrt{\frac{\lambda_j}{g_j}} \sqrt{\frac{r_j(\Sigma)}{n} + \frac{r_j(\Sigma)}{n^{1-2/p}}}$$

Idea of the proof of Fuk-Nagaev inequality

- $X_1, \ldots, X_n$ are centered self-adjoint random matrices such that $\mathbb{E}\|X_1\|^p < \infty$ and $M := \max_j \|X_j\|$.

$$\mathbb{P}\left(\left\|\sum_{j=1}^{n} X_j\right\| \geq t\right) \leq ?$$

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- "Double truncation:" set $U \asymp \mathbb{E}M$, $y = ct$ and

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- Some work is required to prove that

$$\mathbb{P}\left(\left\|\sum_{k=1}^n \Delta_j^{(1)}\right\| > t/16\right) \lesssim \left(\frac{p}{\log(p)}\right)^{2p} \left(\frac{\mathbb{E}M^p}{t^p}\right)^2$$

  Main tools are the Hoffmann-Jørgensen + Talagrand's comparison inequalities between sums and maxima.

Thank you for listening!