Self-normalization of sums of dependent random variables

Thomas Mikosch (Copenhagen),
Olivier Wintenberger (Sorbonne Paris, Pauliinstitut Wien), Muneya Matsui (Nagoya)

Self-normalized sums are often considered to be "more robust" as regards the influence of large obervations. They are very much present in log-return data of stock prices and indices, foreign exchange rates, interest rates, etc.

In this context a stunning result was proved by Logan et al. (1973). Assuming an iid centered sequence (X_t) with a regularly varying tail, i.e., $\mathbb{P}(\pm X_t > x) \sim p_\pm\, x^{-\alpha}$ as $x \to \infty$, the studentized sum $S_n = X_1 + \cdots + X_n$ has asymptotically a Gaussian tail even if $\alpha \in (1,2)$, i.e., when $\mathrm{var}(X_1) = \infty$. This result is in agreement with the fact that S_n and the sample standard deviation are of the same asymptotic order. Moreover, if $\alpha \in (1,2)$, (S_n/a_n) converges in distribution to an α-stable law with a (commonly unknown) normalizing sequence (a_n). Self-normalization avoids knowledge of (a_n). Similar results remain valid for S_n under self-normalization with the maximum of $|X_1|, \ldots, |X_n|$.

If (X_t) is (strictly) stationary and regularly varying (in a sense to be defined) with index $\alpha \in (0,2)$ similar asymptotic theory is valid for self-normalized sums. These include financial time series model such as stochastic volatility models, GARCH processes, autoregressive conditional durations models. A particularly nice case is a regularly varying linear process with infinite variance: limit results for the self-normalized sample mean are the same as in the iid case modulo some scaling constants; see Davis and Resnick (1985). Unfortunately, this is (almost) the only nice case. In presence of extremal clusters in the sequence limit theory for self-normalized sums becomes complicated and depends on the model at hand. In particular, the limit laws may lack moments, in contrast to the iid case, and graphical tools based on self-normalized quantities may fool one; see Matsui et al. (2024) and Mikosch and Wintenberger (2024).

1. Logan, B.F., Mallows, C.L., Rice, S.O. and Shepp, L.A. (1973) Limit distributions of self-normalized sums. *Ann. Probab.* **1**, 788–809.
2. Davis, R.A. and Resnick, S.I. (1985) Limit theory for moving averages of random variables with regularly varying tail probabilities. *Ann. Probab.* **13**, 179–195.
3. Matsui, M, Mikosch, T. and Wintenberger, O. (2024) Self-normalized partial sums of heavy-tailed time series. *arXiv* 2303.17221.
4. Mikosch, T. and Wintenberger, O. (2024) *Extreme Value Theory for Time Series. Models with Power-Law Tails.* Springer, New York.